Entanglement sharing in $E \otimes \epsilon$ Jahn-Teller model in the presence of a magnetic field

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We discuss the ground state entanglement of the $E \otimes \epsilon$ Jahn-Teller model in the presence of a strong transverse magnetic field as a function of the vibronic coupling strength. A complete characterization is given of the phenomenon of entanglement sharing in a system composed by a qubit coupled to two bosonic modes. Using the residual I-tangle, we find that three-partite entanglement is significantly present in the system in the parameter region near the bifurcation point of the corresponding classical model.

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I. INTRODUCTION

Tools developed in the realm of quantum information theory are increasingly being used to investigate fundamental condensed matter problems [1]. In particular, many model-systems exhibiting quantum phase transitions have been explored, and new insights in their behavior has been gained by studying the entanglement [2], the block entropy [3] and the fidelity [4]. In particular, it has been demonstrated in general that, apart from accidental cancellations, entanglement measures always become singular near the critical points [5] (in the thermodynamic limit) and exhibit a scaling behavior (for finite size systems).

Moreover, entanglement has been shown to display not only the signatures of the critical behavior corresponding to quantum phase transitions, but also to signal the presence of bifurcations in the corresponding semiclassical limit [6, 7]. This has been demonstrated, for example, in some spin-boson models in the strong coupling regime, including the Dicke [8] and the Jahn-Teller models [9, 10]. In fact, in the collective Dicke model, the two aspects of quantum phase transition and classical bifurcation have been shown to be related in the adiabatic limit, in which scaling laws have been recently derived for the ground state entanglement [9, 11, 12].

Generically, spin-boson models describe the linear coupling of one [13, 14, 15] or many [16] bosonic modes (typically, photons or phonons) with electronic or pseudo-spin degrees of freedom, usually represented as two level systems (qubits). These models have been used to explore environment induced decoherence and have been shown to display peculiar properties of entanglement [17].

In this paper, we concentrate on one model of this class, the Jahn-Teller (JT) model [18, 19], involving an electron-nuclei system, in which a doubly degenerate electronic state (usually denoted as E) is coupled to a doubly degenerate nuclear displacement mode (ϵ), with the two bosonic modes coupled to different (orthogonal) spin-directions of the qubit. This is one of the most investigated problems in molecular physics for which a variety of interesting quantum properties have been demonstrated, despite the fact that the corresponding Hamiltonian is not exactly solvable. Particularly relevant from our point of view, is Ref. [9], where ground state entanglement has been investigated for this model-system in the presence of a transverse magnetic field, by making use of an approximate analytic form of the ground state and of numerical diagonalization with a truncated basis. There, by studying the von Neumann entropy, it has been shown that the field forces the coupled system into a maximally entangled state in the large coupling limit.

Besides this aspect, the $E\otimes\epsilon$ system is interesting from many respects. Here we concentrate on its multipartite structure. Indeed, the model describes a tripartite system with an Hilbert space structure of the kind $2\otimes\infty\otimes\infty$ for which we are able to discuss the sharing properties of entanglement in the adiabatic limit.

In general, quantifying three-partite entanglement is an extremely difficult task. For the case of qubits, the CKW conjecture [20], recently demonstrated by Osborne and Verstraete [21, 22], offers us the powerful instruments of the monogamy inequality and the residual tangle, which have been already employed to interpret some magnetic behaviors [23]. Related results concerning the monogamy have been achieved in Ref. [24], for the case of continuous variables. However, no general method has been developed for hybrid systems; that is, those including both discrete and continuous variables. These systems are extremely interesting for many information theoretic applications, including the implementation of quantum memories or the possibility of entanglement concentration and purification [25]. In this respect, we think it is interesting to study some relevant case, such as the JT model we face in this paper. In a related work, Tessier et al. [26] examined the case of two-atom Tavis-Cumnmings model, making use of the Osborne formula [27] to obtain the I-tangle.

With these motivations, this paper explores the sharing structure of entanglement of the $E \otimes \epsilon$ JT system in the presence of a strong uniform magnetic field, whose

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presence has been shown to give rise to interesting consequences in connection with the Berry Phase [28]. Our approach is based on the adiabatic procedure which has been already applied to the case of a qubit strongly coupled to a single slow resonator [10].

The paper is organized as follows: In Sec. II we formulate the $E \otimes \epsilon$ model in the presence of a magnetic field and discuss its solution in the adiabatic approximation; in Sec. III various entanglement measures are evaluated, for which some analytic approximations are derived in Sec. IV. Finally, Sec. V summarizes our main findings.

II. THE $E \otimes \epsilon$ MODEL AND ITS SOLUTION IN THE PRESENCE OF AN EXTERNAL FIELD

The standard JT model describes a qubit interacting with two degenerate harmonic modes (conventionally labelled θ and ϵ). The model Hamiltonian in the presence of an external field is the following

$$\begin{aligned} H &= \frac{\omega}{2}\left(p_\theta^2 + p_\epsilon^2 + q_\theta^2 + q_\epsilon^2\right)\sigma_0 \\ &+ \lambda\left(q_\theta \sigma_x + q_\epsilon \sigma_y\right) + \Delta\sigma_z \end{aligned} \tag{1}$$

where we have chosen unit such that $\hbar = c = 1$. Here ω is the natural frequency of the identical oscillators, Δ is the strength of the magnetic field (taken orthogonal to the directions of the couplings) and also represents the qubit transition frequency, λ is the coupling constant, $\sigma_0 = I$, σ_x, σ_y and σ_z are the usual Pauli matrix and (q_θ, q_ϵ) are real normal coordinates of the vibrational modes.

The system is invariant under rotations around the magnetic field axis and thus there is a conserved operator $\hat{J}_z$, such that $[H, \hat{J}_z] = 0$, and which is given by

$$\hat{J}_z = \hat{L}_z \sigma_0 + \frac{1}{2}\sigma_z \tag{2}$$

L_z being the z component of the orbital angular momentum

$$\hat{L}_z = q_\theta p_\epsilon - q_\epsilon p_\theta \tag{3}$$

We will take advantage of this symmetry to employ the eigenvalues of $\hat{J}_z$ as labels of the energy eigenstates.

The ground state of the Hamiltonian will be found in the Born-Oppenheimer approximation under the assumption of a fast qubit, which is easily realized for strong external fields ($\Delta \gg \omega$). The whole procedure can be followed more plainly by rewriting the Hamiltonian (1) in polar coordinates as follows

$$H = \frac{\omega}{2}\left[\left(|\vec{p}|^2 + |\vec{q}|^2\right)\sigma_0 + \vec{\Theta}\cdot\vec{\sigma}\right] \tag{4}$$

with $|\vec{p}|^2 = p_\theta^2 + p_\epsilon^2$, $\quad |\vec{q}|^2 = q_\theta^2 + q_\epsilon^2$, $\quad \phi = \arctan(q_\epsilon/q_\theta)$. Notice that the qubit dynamics is governed by the effective $\vec{q}$-parametrized magnetic field

$$\vec{\Theta} = (Lq\cos\phi, Lq\sin\phi, D) \tag{5}$$

where we have introduced the dimensionless parameters $D = 2\Delta/\omega$ and $L = 2\sqrt{2}\lambda/\omega$.
In the adiabatic assumption of *slow* bosonic modes, and as a first step in the Born-Oppenheimer procedure, we will regard $\vec{\Theta}$ as approximately static and solve the qubit dynamics for fixed $\vec{q}$.

More formally, we look for a solution of the bi-dimensional Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ written in terms of qubit $|\chi(\vec{q})\rangle$ and oscillator $\varphi(\vec{q})$ functions as

$$|\psi\rangle = \int d^2q\, |\psi(\vec{q})\rangle = \int d^2q\, \varphi(\vec{q})|\vec{q}\rangle \otimes |\chi(\vec{q})\rangle \tag{6}$$

where $|\chi(\vec{q})\rangle$ are the eigenstates of the "adiabatic" equation of the qubit part

$$\vec{\Theta}\cdot\vec{\sigma}|\chi_\pm(\vec{q})\rangle = \pm\Theta(q)|\chi_\pm(\vec{q})\rangle\,, \tag{7}$$

which gives the eigenvalues

$$\Theta(q) = |\vec{\Theta}| = \sqrt{D^2 + L^2q^2}\,. \tag{8}$$

The two eigenstates of Eq. (7) are

$$|\chi_-(\vec{q})\rangle = e^{-i\frac{\phi}{2}}a(q)|\uparrow\rangle - e^{i\frac{\phi}{2}}b(q)|\downarrow\rangle \tag{9}$$

$$|\chi_+(\vec{q})\rangle = e^{-i\frac{\phi}{2}}b(q)|\uparrow\rangle + e^{i\frac{\phi}{2}}a(q)|\downarrow\rangle \tag{10}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the ± 1 eigenstates of σ_z, while

$$a(q) = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{D}{\Theta(q)}}, \tag{11}$$

$$b(q) = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{D}{\Theta(q)}} \tag{12}$$

The eigenvalues can be then considered as distortion of the harmonic potential, so that the oscillators are effectively subject to the adiabatic potentials $W_\pm = q^2 \pm \Theta(q)$ when the qubit is in $|\chi_\pm\rangle$.

The problem, then, reduces to find the solution of a bi-dimensional Schrödinger equation with W as the potential energy. This is a difficult task, which can be simplified by exploiting the rotational symmetry.

Since $\hat{J}_z$ commutes with H and due to the functional dependence of the adiabatic qubit eigenstates on the polar angle ϕ, we can factorize the oscillator wave function in the form

$$\varphi(q,\phi) = (2\pi)^{-1/2}\varphi_j(q)e^{ij\phi} \tag{13}$$

where $j = \pm 1/2, \pm 3/2, \ldots$ is the eigenvalue of the operator $\hat{J}_z$.

From Eqs.(9-10) the unitary transformation that diagonalizes the potential energy matrix is obtained as

$$U = \begin{pmatrix} e^{-i\frac{\phi}{2}}b(q) & e^{-i\frac{\phi}{2}}a(q) \\ e^{i\frac{\phi}{2}}a(q) & -e^{i\frac{\phi}{2}}b(q) \end{pmatrix} \tag{14}$$

The transformed Hamiltonian has the form

$$\tilde{H} = U^\dagger H U = \frac{\omega}{2}\left[\left(|\vec{p}|^2 + |\vec{q}|^2\right)\sigma_0 + \Theta(q)\sigma_z + \Lambda(\vec{q})\right] \quad (15)$$

where

$$\Lambda(\vec{q}) = U^\dagger |\vec{p}|^2 U + 2U^\dagger \vec{p} U \cdot \vec{p} = \Lambda_0\sigma_0 + \vec{\Lambda}\cdot\vec{\sigma} \quad (16)$$

The components of rotated effective field Λ are

$$\Lambda_0 = \frac{1}{4}\left(\frac{1}{q^2} + \frac{L^2 D^2}{\Theta^4}\right) \quad (17)$$

$$\Lambda_x = -\frac{L}{q\Theta}\left[\hat{L}_z - \frac{D}{\Theta}\left(\frac{1}{2} - \frac{D^2}{\Theta^2}\right)\right] \quad (18)$$

$$\Lambda_y = -\frac{DL}{\Theta^2}\frac{\partial}{\partial q} \quad (19)$$

and

$$\Lambda_z = \left[-\frac{1}{q^2} + \frac{L^2}{\Theta(\Theta+D)}\right]\hat{L}_z \quad (20)$$

In the absence of magnetic field (the limit $D \to 0$),

$$\Lambda_0 = \frac{1}{4q^2}\,, \quad \Lambda_x = -\frac{1}{q^2}\hat{L}_z\,, \quad \Lambda_y = \Lambda_z = 0 \quad (21)$$

and the well-known result for the linear $E\otimes\epsilon$ Jahn-Teller model is recovered, [29], i.e.

$$\begin{aligned}\tilde{H} &= \frac{\omega}{2}\left[-\left(\frac{\partial^2}{\partial q^2} + \frac{1}{q}\frac{\partial}{\partial q} - q^2\right)\sigma_0 + Lq\sigma_z\right.\\ &+ \left.\frac{1}{q^2}\left(\hat{L}_z\sigma_0 - \frac{\sigma_x}{2}\right)^2\right] \end{aligned}\quad (22)$$

In the strong coupling limit ($L \gg 1$), one can neglect the off-diagonal (non-adiabatic) terms in this expression, so that the factorization (13) leads to a second-order equation for the radial function $\varphi_j(q)$ of two adiabatic potential energy surfaces (APES)

$$\left[-\frac{d^2}{dq^2} - \frac{1}{q}\frac{\partial}{\partial q} + q^2 \pm Lq + \frac{j^2}{q^2} - \varepsilon_j\right]\varphi_j(q) = 0 \quad (23)$$

where the term j^2/q^2 plays the role of the centrifugal energy. In this case, the ground state is characterized by the quantum number $j = \pm 1/2$ and is thus doubly degenerate.

The off-diagonal non adiabatic terms can be neglected directly in Eq.(15) under the assumption of a strong transverse magnetic field, i.e. $D \gg 1$. This is the regime we will discuss. For comparison, in this limit, the Hamiltonian (15) becomes

$$\begin{aligned}\tilde{H} &= \frac{\omega}{2}\left[-\left(\frac{\partial^2}{\partial q^2} + \frac{1}{q}\frac{\partial}{\partial q} - q^2\right)\sigma_0 + \Theta\sigma_z\right.\\ &+ \left.\frac{1}{q^2}\left(\hat{L}_z\sigma_0 - \frac{\sigma_z}{2}\right)^2\right] \end{aligned}\quad (24)$$

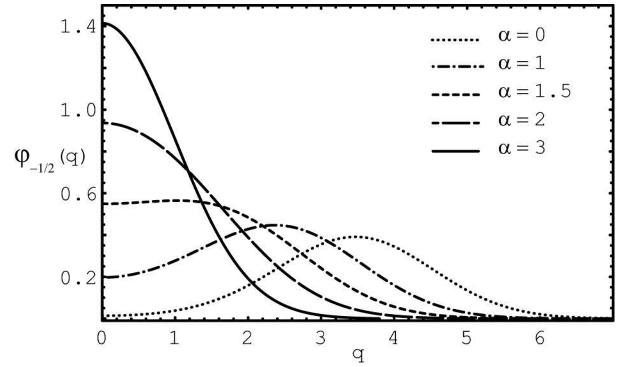

FIG. 1: Normalized ground state wave function for the oscillators in the lower adiabatic potential, for $D = 10$ and different values of α.

The factorization (13) leads to a different equation for the radial function $\varphi_j(q)$

$$\left[-\frac{d^2}{dq^2} - \frac{1}{q}\frac{\partial}{\partial q} + q^2 \pm \Theta + \frac{1}{q^2}\left(j \mp \frac{1}{2}\right)^2 - \varepsilon_j\right]\varphi_j(q) = 0 \quad (25)$$

with the result that, in the presence of a magnetic field, the degeneracy present in the linear JT model is broken.

When $D \gg 1$ the motion will remain on the lowest Adiabatic Potential Energy Surface (APES) given by $W_- = q^2 - \Theta(q)$ and characterized by the quantum number $j = -1/2$ (notice that this implies that the centrifugal energy equals zero).

Introducing the dimensionless parameter $\alpha = L^2/2D$, one can show that for $\alpha \le 1$, the potential $W_-(q)$ is just a broadened harmonic potential surface with a minimum at $q = 0$ and $W_-(0) = -D$. For $\alpha > 1$, on the other hand, the coupling of the oscillator with the qubit splits the lowest APES producing a double-well potential surface with (a circle of) minima at

$$q = q_0 = \sqrt{\frac{D}{2}\left(\alpha - \frac{1}{\alpha}\right)}\,, \quad (26)$$

with

$$W_-(q_0) = -\frac{D}{2}\left(\alpha + \frac{1}{\alpha}\right)\,. \quad (27)$$

In Fig.(1), the ground state wave function $\varphi_{-1/2}(q)$ is shown for $D = 10$ and different values of α. We can see that the maximum probability amplitude is always found around q_0, and that, as α decreases, this moves far and far away from the origin.

III. GROUND STATE ENTANGLEMENT

The expression of the ground state obtained in the previous section enables us to compute the entanglement

content of the system. We have three independent subsystems: the qubit, the radial, and the azimuthal degrees of freedom in which we have decomposed the two oscillators (from now on, we indicate these subsystems with the labels E, q and ϕ, respectively).

In this section we will evaluate the amount of entanglement for every possible bi-partition and then use the monogamy inequality to obtain the residual tangle. First, however, we briefly review the formalism employed.

A. I-Tangle formalism

To quantify the entanglement for each of the bi-partitions of the model we will make use of the I-tangle [30], which for a rank-2 mixed state ρ_{AB} can be explicitly evaluated as, [27],

$$\tau(\rho_{AB}) = \mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB}) + 2\lambda_{min}^{(AB)}\left[1 - \mathrm{Tr}(\rho_{AB}^2)\right] \quad (28)$$

where $\tilde{\rho}_{AB}$ is the result of the action of the *universal state inverter* [30] on ρ_{AB}

$$\tilde{\rho}_{AB} = S_A \otimes S_B(\rho_{AB}) \quad (29)$$

and $\lambda_{min}^{(AB)}$ is the smallest eigenvalue of the M matrix defined by Osborne [27] which is defined and then evaluated for our case in appendix A.

The universal inverter S_i is defined to map every pure state $\rho_i = |\psi\rangle\langle\psi|$ into a positive multiple of its orthogonal projector, i.e. $S_i(\rho_i) = \nu_i\,(I - \rho_i)$. For an arbitrary operator O, it gives

$$S_i(O) = \nu_i\,[\mathrm{Tr}(O)I - O] \quad (30)$$

where ν_i is an arbitrary real constant (which we choose to be unit). The tensor product in Eq.(29), applied to an arbitrary joint density operator ρ_{AB}, is given by

$$S_A \otimes S_B(\rho_{AB}) = I_A \otimes I_B - \rho_A \otimes I_B - I_A \otimes \rho_B + \rho_{AB} \quad (31)$$

where ρ_A and ρ_B are the reduced density operators obtained from ρ_{AB}. Putting everything together,

$$\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB}) = 1 - \mathrm{Tr}(\rho_A^2) - \mathrm{Tr}(\rho_B^2) + \mathrm{Tr}(\rho_{AB}^2) \quad (32)$$

For a joint pure state ($\mathrm{Tr}(\rho_{AB}^2) = \mathrm{Tr}(\rho_{AB}) = 1$) the I-tangle (28) becomes

$$\tau_{AB} = 2\left[1 - \mathrm{Tr}(\rho_A^2)\right] \quad (33)$$

where $\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(\rho_B^2)$.

We will employ relations (28) and (33) several times in the following.

B. Ground state density operators

In our case the ground state density operator takes the form

$$\rho = \int d^2q d^2q' \varphi_{-1/2}(\vec{q})\varphi_{-1/2}^*(\vec{q}\,')|\vec{q}\rangle\langle\vec{q}\,'|\left(|\chi_-(\vec{q})\rangle\langle\chi_-(\vec{q}\,')|\right) \quad (34)$$

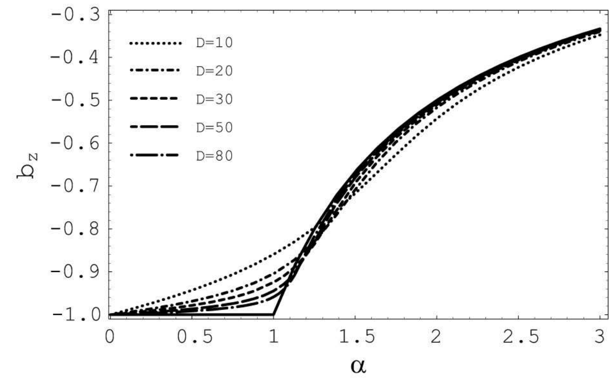

FIG. 2: The dependence of the ground state expectation value $b_z = \langle\sigma_z\rangle$ as a function of the parameter α, for various values of D. The solid line corresponds to $D \to \infty$.

There are six nonequivalent bi-partitions: (i) qubit-oscillators $E \otimes (\phi\, q)$; (ii) angular degree of freedom-remainder $\phi \otimes (E\, q)$; (iii) radial degree of freedom-remainder $q \otimes (E\,\phi)$; (iv) angular degree of freedom-qubit $\phi \otimes E$; (v) radial degree of freedom-qubit $q \otimes E$; (vi) radial degree of freedom-angular degree of freedom $\phi \otimes q$.

To start evaluating the various tangles, it is useful to re-write the ground state density operator (34) as

$$\begin{aligned}\rho \;=\; & |a\rangle|f_1\rangle|\uparrow\rangle\langle\uparrow|\langle f_1|\langle a| + |b\rangle|f_2\rangle|\downarrow\rangle\langle\downarrow|\langle f_2|\langle b| \\ - \; & |a\rangle|f_1\rangle|\uparrow\rangle\langle\downarrow|\langle f_2|\langle b| - |b\rangle|f_2\rangle|\downarrow\rangle\langle\uparrow|\langle f_1|\langle a|\end{aligned} \quad (35)$$

where

$$|a\rangle \;=\; \int_0^\infty dq\, q\, \varphi_{-1/2}(q)\; a(q)\,|q\rangle\,, \quad (36)$$

$$|b\rangle \;=\; \int_0^\infty dq\, q\, \varphi_{-1/2}(q)\; b(q)\,|q\rangle \quad (37)$$

are two (non normalized) states of the q-mode, while $|f_i\rangle\; i = 1, 2$ are the two relevant (and ortho-normal) states of the angular degree of freedom:

$$|f_1\rangle = \int_0^{2\pi} \frac{d\phi}{\sqrt{2\pi}} e^{-i\phi}|\phi\rangle\,,\; |f_2\rangle = \int_0^{2\pi} \frac{d\phi}{\sqrt{2\pi}}|\phi\rangle \quad (38)$$

The situation is similar to that described in Ref. [9]: the angular degree of freedom is constrained to a two-dimensional subspace of its total Hilbert space and our tripartite system can be considered as a $2\otimes 2\otimes\infty$ system.

For the set of states (36-37) we have:

$$\langle a|a\rangle = \frac{1 - b_z}{2}\,,\; \langle b|b\rangle = \frac{1 + b_z}{2} \quad (39)$$

where

$$b_z = -\int_0^\infty q\varphi_0^2(q)\frac{D}{\Theta(q)}dq\,, \quad (40)$$

is the z-component of the Bloch vector $\vec{b} = \langle\vec{\sigma}\rangle$ and

$$\langle a|b\rangle = \langle b|a\rangle = \int_0^\infty q\varphi_0^2(q)\frac{Lq}{\Theta(q)}dq = -b_\phi \quad (41)$$

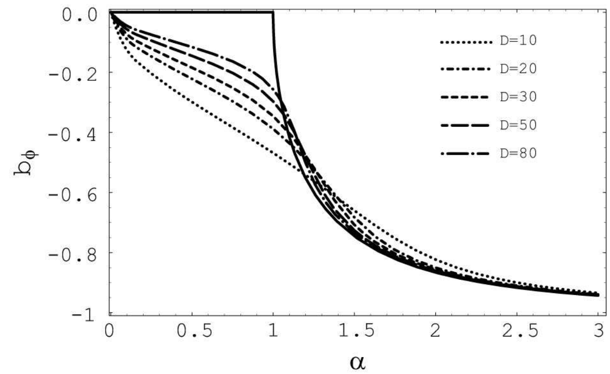

FIG. 3: The equatorial component of the Bloch vector along the ϕ direction, $b_\phi = \langle \cos\phi\,\sigma_x + \sin\phi\,\sigma_y \rangle$ shown as a function of α, for different values of D. The solid line corresponds to $D \to \infty$.

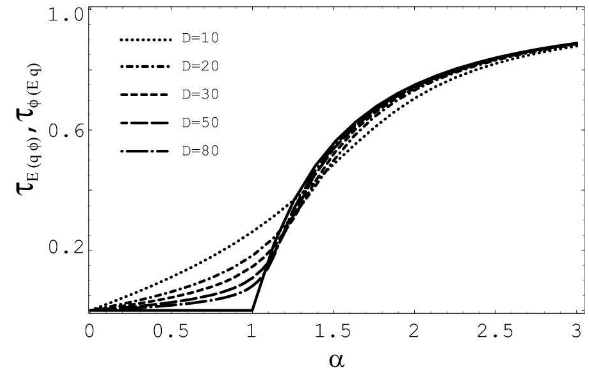

FIG. 4: The tangle between the qubit and the oscillators and between the angular degree of freedom and the rest as a function of the interaction strength as measured by α for different values of D.

where $b_\phi = \langle \cos\phi\,\sigma_x + \sin\phi\,\sigma_y \rangle$ is the equatorial component in the ϕ direction.

In Figs.(2) and (3), we show the dependence of the ground state expectation values b_z and b_ϕ on the dimensionless quantity α for various values of the external field D (broken lines). The continuous plot describes the case of very large field ($D \to \infty$) for which an analytic expression is obtained in section IV. We will see in the following that these two parameters completely characterize the ground state.

From the plots, one can see that for small interaction strengths (that is, small α's) the external field dominates and forces the qubit state along its direction; indeed, $b_z \simeq -1$ and $b_\phi \simeq 0$. On the other hand, for a large enough α the qubit is strongly correlated with the angular mode ϕ (loosely speaking, it is 'oriented' along ϕ) with a small residual polarization along the magnetic field. At $\alpha = 1$ a singular behavior is found for very large fields, that is analyzed below.

From eq. (35), the marginal density operators are easily obtained. For the partitions $\phi \otimes q$ and $E \otimes q$ one has:

$$\rho_{\phi q} = \sum_{S=\uparrow,\downarrow} \langle s|\rho|s\rangle = |a\rangle|f_1\rangle\langle f_1|\langle a| + |b\rangle|f_2\rangle\langle f_2|\langle b| \quad (42)$$

$$\rho_{Eq} = \sum_{i=1,2} \langle f_i|\rho|f_i\rangle = |a\rangle|\uparrow\rangle\langle\uparrow|\langle a| + |b\rangle|\downarrow\rangle\langle\downarrow|\langle b| \quad (43)$$

Tracing over q gives a state for $E \otimes \phi$ which has a bit more involved expression:

$$\begin{aligned} \rho_{E\phi} &= \int_0^\infty q\langle q|\rho|q\rangle dq \\ &= \frac{1+b_z}{2}|f_1\rangle|\uparrow\rangle\langle\uparrow|\langle f_1| + \frac{1-b_z}{2}|f_2\rangle|\downarrow\rangle\langle\downarrow|\langle f_2| \\ &+ \frac{b_\phi}{2}(|f_1\rangle|\uparrow\rangle\langle\downarrow|\langle f_2| + |f_2\rangle|\downarrow\rangle\langle\uparrow|\langle f_1|) \end{aligned} \quad (44)$$

As stated above, the reduced density operators are completely specified by the three set of states introduced above for the various sub-systems, and by the parameters b_z and b_ϕ.

C. Qubit-oscillators, ϕ-remainder and q-remainder tangles

In this sub-section, we evaluate the entanglement of each one of the three subsystems with the remainder. Since the overall state is pure, the procedure is quite straightforward. The tangle of the qubit with the two oscillators is

$$\tau_{E(\phi q)} = 2\left[1 - \mathrm{Tr}(\rho_E^2)\right] = 1 - b_z^2 \quad (45)$$

The tangle between the angular degree of freedom with the rest of the system is

$$\tau_{\phi(Eq)} = 2\left[1 - \mathrm{Tr}(\rho_\phi^2)\right] . \quad (46)$$

Its expression coincides with $\tau_{E(\phi q)}$ since the marginal density operator for the ϕ degree of freedom has the same non-zero entries of the qubit one

$$\rho_\phi = \frac{1+b_z}{2}|f_1\rangle\langle f_1| + \frac{1-b_z}{2}|f_2\rangle\langle f_2| \quad (47)$$

These two tangles are shown in Fig. (4), where it can be seen that the qubit (as well as the ϕ sub-system) essentially factorizes for small interaction strengths. This is more and more true for increasing external field and is due to the fact that the field itself keeps the spin aligned, despite its interaction with the oscillators. For values of α larger than 1, the interaction dominates more and more. This implies that qubit and angular degree of freedom becomes more and more entangled; indeed, the tangles saturate to 1 for large enough α's.

To be more precise, and as better discussed below, the ground state contains (for almost every α) essentially bipartite entanglement as these two degrees of freedom correlate to each other, with very little involvement of the q part.

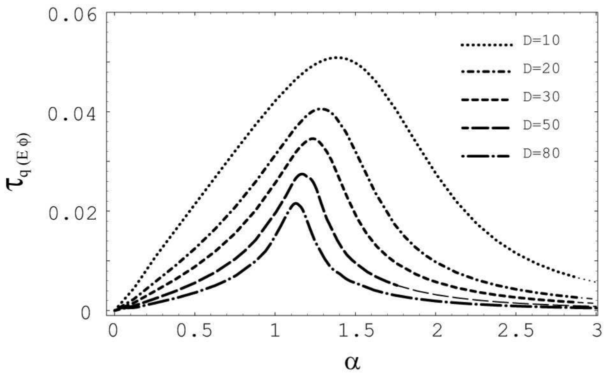

FIG. 5: The tangle between between the radial degree of freedom and the remainder as a function of α for different values of D. Notice that it is notably different from zero only around $\alpha = 1$.

To see that this is indeed the case, we start by evaluating the entanglement to which the radial degree of freedom participates. The tangle $\tau_{q(E\phi)}$ is given by

$$\tau_{q(E\phi)} = 2\left[1 - \mathrm{Tr}(\rho_q^2)\right] \equiv 1 - b_z^2 - b_\phi^2 \tag{48}$$

This function is shown in Fig. (5), where one can see that the radial degree of freedom is very poorly correlated with the others. This situation is reminiscent of the one obtained when a qubit interacts with a single oscillator mode in the presence of a tilted external field which gives rise to an "asymmetry" in the adiabatic potential, see [10]. The ϕ-mode, here, plays exactly the same role of such an asymmetry. In fact, it destroys the correlations between the radial mode and the qubit due to the monogamy of entanglement.

It is noteworthy, however, that the entanglement between E and q is more relevant in the region around $\alpha = 1$. Indeed, the maximum of $\tau_{q(E\phi)}$ moves towards this point as the field increases and this is exactly the point where, in the strict adiabatic limit of very large D, the tangle becomes discontinuous.

We show in the following sections that this is precisely the region in which a true three-partite entanglement (as measured by the residual tangle) is present. To evaluate the three-partite correlations, however, we first need to evaluate entanglement for the other possible bi-partitions in which one of the three subsystems is traced out. This can be done explicitly thanks to the Osborne method reviewed above.

D. Angular degree of freedom-qubit tangle

After tracing over the radial mode q, the reduced density operator for the partition $E \otimes \phi$, eq. (44), can be re-written in the form

$$\rho_{E\phi} = \frac{1 + \sqrt{b_z^2 + b_\phi^2}}{2} |v_1\rangle\langle v_1| + \frac{1 - \sqrt{b_z^2 + b_\phi^2}}{2} |v_2\rangle\langle v_2| \tag{49}$$

where

$$|v_1\rangle = \beta_1 |f_1\rangle|\uparrow\rangle + \beta_2 |f_2\rangle|\downarrow\rangle \tag{50}$$

$$|v_2\rangle = \gamma_1 |f_1\rangle|\uparrow\rangle - \gamma_2 |f_2\rangle|\downarrow\rangle \tag{51}$$

with

$$\beta_1 = \left[1 + \left(\frac{b_z}{b_\phi} + \sqrt{1 + \frac{b_z^2}{b_\phi^2}}\right)^2\right]^{-1/2}, \ \beta_2 = \sqrt{1 - \beta_1^2} \tag{52}$$

and

$$\gamma_1 = \left[1 + \left(-\frac{b_z}{b_\phi} + \sqrt{1 + \frac{b_z^2}{b_\phi^2}}\right)^2\right]^{-1/2}, \ \gamma_2 = \sqrt{1 - \gamma_1^2} \tag{53}$$

The vectors $|v_i\rangle, i = 1,2$ are the only eigen-kets of $\rho_{E\phi}$ with non-zero eigenvalues given by $r_i = \left(1 \pm \sqrt{b_z^2 + b_\phi^2}\right)/2$.

This form (which, by the way, shows that the matrix has rank two) is particularly useful to apply the Osborne procedure. A straightforward calculation gives the tangle in the form

$$\tau_{E\phi} = \frac{1 - b_z^2}{2}\left(1 + 2\lambda_{min}^{(E\phi)}\right) + \frac{b_\phi^2}{2}\left(1 - 2\lambda_{min}^{(E\phi)}\right) \tag{54}$$

where $\lambda_{min}^{(E\phi)}$ is obtained in appendix A

$$\lambda_{min}^{(E\phi)} = \frac{1}{4}\left(1 - \sqrt{1 + \frac{8b_z^2}{b_z^2 + b_\phi^2}}\right) \tag{55}$$

E. q-ϕ and q-E tangles

The two remaining bi-partitions of the system are those consisting of the radial degree of freedom and either the angular mode or the qubit. These turn out to have no entanglement at all. Indeed, one has:

$$\mathrm{Tr}(\rho_{Eq}\tilde{\rho}_{Eq}) = \mathrm{Tr}(\rho_{\phi q}\tilde{\rho}_{\phi q}) = \frac{1 - b_z^2 - b_\phi^2}{2} \tag{56}$$

$$\lambda_{min}^{(Eq)} = \lambda_{min}^{(\phi q)} = -\frac{1 - b_z^2 - b_\phi^2}{2(1 - b_z^2)} \tag{57}$$

Putting everything together in eq. (28), one has

$$\tau_{Eq} = \tau_{\phi q} = 0 \tag{58}$$

F. Residual tangle

The amount of entanglement for the various bi-partitions that we have evaluated above, do not give by

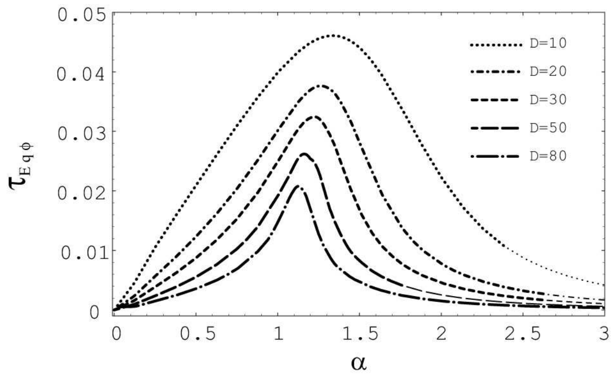

FIG. 6: The I-residual tangle given in eq. (64), shown for different values of the external magnetic field.

themselves any indication neither on the sharing properties nor on the global, three-partite quantum correlations. Coffman et al. [20] have explored this problem in a system of three qubits and introduced a quantity known as the residual tangle, to describe the collective entanglement content of a state:

$$\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{AC} \tag{59}$$

When subsystems A, B and C are entangled with each other, the tangle of A with B plus the tangle of A with C cannot exceed the tangle of A with the joint subsystem BC. This result has been proved valid for any multipartite state of qubits [21].

In the $E\otimes\epsilon$ JT model, we cannot simply use the definition (59) of the residual tangle since our three subsystems no longer have equal Hilbert space dimension and symmetry under permutations of the subsystems, which is present in eq. (59) would be lost.

Tessier et al. [26] have faced a similar problem and proposed to generalize the quantity (59) by just taking the average of the three residual tangles to introduce the *I-residual tangle* which has, by definition, the desired permutation invariance:

$$\tau_{E\phi q} = \frac{1}{3}\left[\tau^{(1)}_{E\phi q} + \tau^{(2)}_{E\phi q} + \tau^{(3)}_{E\phi q}\right] \tag{60}$$

where

$$\tau^{(1)}_{E\phi q} = \tau_{E(\phi q)} - \tau_{E\phi} - \tau_{Eq} \tag{61}$$

$$\tau^{(2)}_{E\phi q} = \tau_{\phi(Eq)} - \tau_{E\phi} - \tau_{\phi q} \tag{62}$$

$$\tau^{(3)}_{E\phi q} = \tau_{q(E\phi)} - \tau_{Eq} - \tau_{\phi q} \tag{63}$$

In our case, $\tau^{(1)}_{E\phi q} = \tau^{(2)}_{E\phi q} \neq \tau^{(3)}_{E\phi q}$, and one easily obtains

$$\tau_{E\phi q} = \frac{2}{3}\tau_{q(E\phi)}\left(1 - \lambda^{(E\phi)}_{min}\right) \tag{64}$$

This quantity is shown in Fig. (6), from which the similarity with the plots of Fig. (5) can be easily grasped. This is due to the fact that the q mode is only involved in genuinely three-partite entanglement as it does not present any bi-partite quantum correlation neither with the qubit nor with the angular mode taken alone.

Again, we notice that the residual tangle is present only within a small region around $\alpha = 1$.

IV. ASYMPTOTIC BEHAVIOR OF THE ENTANGLEMENT

In order to obtain an analytic estimation of the physical quantities evaluated above and for the various entanglement measures introduced, we would need an expression for the ground state wave function $\varphi_{-1/2}(q)$. It is possible to obtain analytically this function under some reasonable approximation for the effective adiabatic potential. In the following we report three distinct approximations, valid in the regimes of *i*) small coupling, $\alpha \ll 1$; *ii*) very large coupling $\alpha \gg 1$; and *iii*) around the cross-over value $\alpha \approx 1$.

A. Small coupling regime

For $\alpha \ll 1$ the adiabatic potential Θ in the Schrödinger equation (25) is approximately harmonic, and the main effect of the qubit is to re-normalize the value of the oscillator frequency by a factor $k = \sqrt{1-\alpha}$. As a result, in this regime the adiabatic ground state wave function for the oscillator is well approximated by the gaussian

$$\varphi_{-1/2}(q) = (2k)^{\frac{1}{2}}\ \exp\left\{-\frac{k}{2}\,q^2\right\}. \tag{65}$$

By repeating the various steps of the previous section, one can obtain approximate expressions for the various tangles introduced above, valid to first order in α. For example,

$$\tau_{E(\phi q)} \equiv \tau_{\phi(Eq)} \simeq \frac{2\alpha}{D}\ , \tau_{E\phi} \simeq \frac{\pi\alpha}{2D} \tag{66}$$

$$\tau_{Eq\phi} \equiv \tau_{q(E\phi)} \simeq \left(2 - \frac{\pi}{2}\right)\frac{\alpha}{D} \tag{67}$$

which we checked to be in very good agreement with the numerical solution given above, and which describe the start-up of entanglement as soon as the interaction is switched on. The last equation shows that (to first order in α), the radial mode is involved only in three-partite entanglement.

B. Strong coupling regime

For $\alpha \gg 1$, the lowest eigenstate should be localized at the minimum of the lowest potential surface. Therefore, by expanding the potential around this minimum [the q_0 of Eq.(26)] and by retaining up to second order

terms, the Schrödinger equation for the lowest sheet can be viewed as the equation for a bi-dimensional shifted harmonic oscillator.

Letting $\tilde{q} = q - q_0$, to be the distance from the minimum, the approximate adiabatic equation for the ground state with $j = -1/2$, becomes

$$\left[\frac{d^2}{d\tilde{q}^2} + \frac{1}{q_0}\left(1 - \frac{\tilde{q}}{q_0}\right)\frac{d}{d\tilde{q}} + v_0 - \kappa^2\tilde{q}^2 + \varepsilon_{-1/2}\right]\varphi_{-1/2}(\tilde{q}) = 0 \tag{68}$$

where $v_0 = \frac{D}{2\alpha}(\alpha^2 - 1)$ is an energy shift, and

$$\kappa = \left(1 - \frac{1}{\alpha^2}\right)^{1/2} \simeq 1 - \frac{1}{2\alpha^2}$$

is, again, a renormalization factor for the oscillator frequency.

To obtain analytic estimates for large α, we can take as an approximate adiabatic ground state for the oscillator the wave function

$$\varphi_{-1/2}(\tilde{q}) \simeq \left(\frac{\kappa q_0^2}{\pi}\right)^{1/4} \exp\left\{-\frac{\kappa}{2}\tilde{q}^2\right\} . \tag{69}$$

In this regime, we have an almost complete quantum correlation between the qubit and the ϕ mode:

$$\tau_{E(\phi q)} = \tau_{\phi(Eq)} = \tau_{E\phi} \simeq 1 - \frac{1}{\alpha^2} , \tag{70}$$

On the other hand, the q mode is almost factorized since its wave function is very localized. As a result, the residual tangle is very close to zero (the leading contribution being of third order in $1/\alpha$):

$$\tau_{q(E\phi)} \simeq \frac{1}{\alpha^3 D} , \qquad \tau_{Eq\phi} \simeq \frac{2}{3\alpha^3 D} \tag{71}$$

C. Critical region

The coupling value corresponding to $\alpha = 1$ divides an essentially separable regime from an entangled one. This point corresponds to a bifurcation in the appropriate semiclassical analogue [9], and we have shown that the region of parameters around $\alpha = 1$ is the only one with non negligible residual tangle. We have also shown that, when the magnetic field increases, this cross-over becomes more and more sharp until a singular behavior is found in the z-magnetization and (as a consequence) in the entanglement measures.

In this section we seek for an analytic description of the system in this parameter region and show that a scaling behavior is found with respect to D. For this reason we call this a critical region.

Above, we have defined the adiabatic potential as

$$W_-(q) = q^2 - \Theta(q) \equiv q^2 - \sqrt{D^2 + L^2 q^2} .$$

For $\alpha \sim 1$ it can be approximated with the quartic expression

$$W_-(q) \simeq -D + (1-\alpha)q^2 + \frac{\alpha^2}{2D}q^4 \tag{72}$$

that describes an anharmonic oscillator for $\alpha \leq 1$, whereas, for $\alpha \geq 1$, it is a double-well potential. As in the single oscillator case [10], this implies that a crossover between a localized state and a Schrödinger cat-like state is obtained. This, in turn, implies a drastic change in the behavior of entanglement.

This approximate potential apparently depends on the two independent parameters α and D, but a reduction to a single-parametric problem can be obtained with the help of Symanzik scaling [31]. This is done by re-casting the Schrödinger equation (always written for $j = -1/2$, see section II), into the equivalent form

$$\left[-\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + \zeta x^2 + x^4\right]\varphi_{-1/2}(x;\zeta) = e_g(\zeta)\,\varphi_{-1/2}(x;\zeta) \tag{73}$$

where $x = q(\alpha^2/2D)^{1/6}$ is a scaled variable. The only remaining scale parameter is, then, $\zeta = \left(2D/\alpha^2\right)^{2/3}(1-\alpha)$, while the ground-state energy is rewritten as

$$\varepsilon_g \equiv \varepsilon_{-1/2} = -D + \left(\frac{\alpha^2}{2D}\right)^{1/3} e_g(\zeta) \tag{74}$$

It can be shown that all of the qubits and oscillator expectation values can be expressed in terms of the diagonal moments:

$$\langle q^\nu \rangle = \int_0^\infty q^{\nu+1}\varphi_{-1/2}^2(q)dq = \left(\frac{2D}{\alpha^2}\right)^{\nu/6}\langle x^\nu\rangle , \tag{75}$$

where

$$\langle x^\nu \rangle = \int_0^\infty x^{\nu+1}\varphi_{-1/2}^2(x;\zeta)dx \tag{76}$$

In fact, the parameter ζ is very small for $\alpha \approx 1$ and we can obtain analytic approximations for every physical quantity we need, by retaining only the first orders of their Taylor expansion in ζ.

For example, the two relevant components of the Bloch vector of the qubit, taken *i*) along the external field (b_z), and *ii*) in the equatorial plane along the ϕ direction (b_ϕ), have the approximate expressions

$$b_z \simeq -1 + \left(\frac{2\alpha}{D^2}\right)^{1/3}\langle x^2\rangle - \frac{3}{2}\left(\frac{2\alpha}{D^2}\right)^{2/3}\langle x^4\rangle \tag{77}$$

$$b_\phi \simeq -\sqrt{2}\left[\left(\frac{2\alpha}{D^2}\right)^{1/6}\langle x\rangle - \left(\frac{2\alpha}{D^2}\right)^{1/2}\langle x^3\rangle\right] . \tag{78}$$

These forms for the components of $\vec{b}$ can be plugged in the general relations for the various tangles obtained in section III, to get

$$\tau_{E(\phi q)} = \tau_{\phi(Eq)} \simeq \left(\frac{4}{D}\right)^{2/3}\langle x^2\rangle , \tag{79}$$

$$\tau_{E\phi} \simeq \left(\frac{4}{D}\right)^{2/3} \langle x \rangle^2 \tag{80}$$

and

$$\tau_{Eq\phi} \equiv \tau_{q(E\phi)} \simeq \left(\frac{4}{D}\right)^{2/3} \left(\langle x^2 \rangle - \langle x \rangle^2\right) \tag{81}$$

All the quantities can be evaluated explicitly once we know the various moments of the scaled position x at $\zeta = 0$. These, however, are just constant numerical values, so that the physical dependence on D and α can be already read from the formula above. In particular, a power-law behavior is found, and both the bi-partite and the residual tangles become singular as $D^{-2/3}$.

For completeness, we give the numerical values of the first moments of the scaled position which are involved in the formula above. For $\alpha = 1$, the problem is reduced to the bi-dimensional motion in a pure quartic potential

$$\left(-\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx} + x^4\right)\varphi_{-1/2}(x;0) = e_{-1/2}(0)\varphi_{-1/2}(x;0) \tag{82}$$

whose energy and all of the moments can be computed numerically. One obtain $e_{-1/2}(0) \simeq 2.3448$, $\langle x \rangle \simeq 0.72737$ and $\langle x^2 \rangle \simeq 0.6515$.

By using these numerical values in Eqs. (79), (80), and (81), we obtain that the scaling with D of the various tangles at $\alpha = 1$ is essentially indistinguishable from the numerical behaviors for large enough fields (i.e. as long as $D > 10$).

V. SUMMARY

We have discussed the sharing structure of entanglement in $E \otimes \epsilon$ JT model in the presence of a strong external field. Using an average residual I-tangle obtained from the monogamy inequality, we have shown that three-partite correlations are important near the point in parameter space that corresponds to the bifurcation of the corresponding classical system. This point divides a separable from an entangled region, and a singular behavior of entanglement is obtained in the strict adiabatic limit. By a detailed analysis performed near this point, we have derived a scaling behavior with respect to the external magnetic field and identified its "critical" exponent.

APPENDIX A: OSBORNE M MATRIX

The central ingredient required for the computation of the I-tangle in Eq.(28) is the real symmetric 3×3 matrix M_{ij}, derived by Osborne in Ref. [27], for a density operator ρ expressed as a convex combination of its eigenvectors:

$$\rho = p|v_1\rangle\langle v_1| + (1-p)|v_2\rangle\langle v_2| \tag{A1}$$

The independent matrix elements of M are constructed in terms of the tensor

$$\begin{aligned} T_{ijkl} &= \mathrm{Tr}(\gamma_{ij}\tilde{\gamma}_{kl}) \\ &= \mathrm{Tr}(\gamma_{ij})\mathrm{Tr}(\gamma_{kl}) - \mathrm{Tr}_A(\mathrm{Tr}_B(\gamma_{ij})\mathrm{Tr}_B(\gamma_{kl})) \\ &- \mathrm{Tr}_B(\mathrm{Tr}_A(\gamma_{ij})\mathrm{Tr}_A(\gamma_{kl})) + \mathrm{Tr}(\gamma_{ij}\gamma_{kl}) \end{aligned} \tag{A2}$$

where $\gamma_{ij} = |v_i\rangle\langle v_j|$. For the partition $E \otimes \phi$ one obtains

$$\begin{aligned} T_{1111} &= 4\beta_1^2\beta_2^2\,, \\ T_{1112} &= T_{1121} = -2\left(\beta_1^3\gamma_1 - \beta_2^3\gamma_2\right)\,, \\ T_{1122} &= T_{2211} = T_{1221} = T_{2112} = 1 - 2\left(\beta_1^2\gamma_1^2 + \beta_2^2\gamma_2^2\right) \\ T_{1222} &= T_{2122} = -2\left(\beta_1\gamma_1^3 - \beta_2\gamma_2^3\right)\,, \\ T_{2222} &= 4\gamma_1^2\gamma_2^2\,. \end{aligned} \tag{A3}$$

from which, using Eqs. (52) and (53), we obtain that the only non-zero matrix elements are

$$\begin{aligned} M_{11} &= \frac{b_z^2}{b_z^2 + b_\phi^2}\,, \\ M_{13} &= M_{31} = \frac{b_z b_\phi}{b_z^2 + b_\phi^2}\,, \\ M_{33} &= \frac{b_\phi^2 - b_z^2}{b_z^2 + b_\phi^2} \end{aligned} \tag{A4}$$

The eigenvalues of this M matrix are thus

$$\lambda_\pm^{(E\phi)} = \frac{1}{4}\left(1 \pm \sqrt{1 + \frac{8b_z^2}{b_z^2 + b_\phi^2}}\right) \tag{A5}$$

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